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INVARIANTS OF 2×2 MATRICES, IRREDUCIBLE $SL(2,\mathbb{C})$ CHARACTERS AND THE MAGNUS TRACE MAP

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ABSTRACT. We obtain an explicit characterization of the stable points of the action of $G = SL(2,\mathbb{C})$ on the cartesian product $G^{\times n}$ by simultaneous conjugation on each factor, in terms of the corresponding invariant functions, and derive from it a simple criterion for irreducibility of representations of finitely generated groups into G. We also obtain analogous results for the action of $SL(2,\mathbb{C})$ on the vector space of n-tuples of 2×2 complex matrices. For a free group F_n of rank n, we show how to generically reconstruct the 2^{n-2} conjugacy classes of representations $F_n \to G$ from their values under the map $T_n : G^{\times n} \cong Hom(F_n, G) \to \mathbb{C}^{3n-3}$ considered in [M], defined by certain $3n-3$ traces of words of length one and two.

1. Introduction and main results

Representation varieties and character varieties of finitely generated groups have been extensively studied in the last three decades, not only for their many interesting properties, but also in relation to subjects such as knot theory and spectral geometry of hyperbolic manifolds, among several others (see for example [CS, Go, H, M] and references therein).

Here, we mainly concentrate on two problems related to the variety of conjugacy classes of representations of finitely generated groups into $G = SL(2,\mathbb{C})$, and particularly the case of representations of a free group. The first is the characterization of the stable points (in the sense of geometric invariant theory) of the action of G on the cartesian product $G^{\times n}$ under simultaneous conjugation, in terms of the corresponding invariant functions. As a consequence, we obtain a simple numerical condition for the irreducibility of representations of finitely generated groups. The second is a detailed study of a map considered by Magnus [M], following earlier work by Vogt [V] and Fricke and Klein [FK], which is related to the question of finding a minimal number of invariant functions on $G^{\times n}$ required to get all the other invariants by algebraic extensions.

We now describe the main results. Let G be the algebraic Lie group $SL(2,\mathbb{C})$ and, for a fixed integer $n \geq 1$, let X_n denote the cartesian product $G^{\times n}$. We are interested in the orbit space of the action of G on the affine variety X_n under simultaneous conjugation on every factor. This is equivalent to the space of conjugacy classes of $SL(2,\mathbb{C})$ representations $\rho : F_n \to G$ of a free group F_n on n elements, since

$$Hom(F_n, G) \cong X_n$$

by fixing a choice of generators of F_n.

In the context of algebraic geometry, we can consider the affine quotient $X_n /\!\!/ G$, whose coordinate ring is the ring $\mathbb{C}[X_n]^G$ of regular functions on X_n that are invariant under the action of G. This is a categorical quotient where the geometric points parametrize closed orbits. It is a consequence of a very general result, the first fundamental theorem of invariants of $m \times m$ matrices (see [P] for $m \geq 2$, or Thm. 2.1 below for $m = 2$), that the building blocks of these G-invariant functions are the following *trace functions*. To any given word $w \in F_n$, the corresponding trace function t_w (sometimes called Fricke character) is the algebraic G-invariant function

$$t_w : Hom(F_n, G) \to \mathbb{C} \tag{1.1}$$

that sends a representation ρ to the trace of the $SL(2,\mathbb{C})$ matrix $\rho(w)$. Moreover, it is a very old result of Vogt and Fricke (see [V] and also [H]) that the ring of these trace functions is finitely generated. One of these finiteness results is as follows. Let $\epsilon_1, ..., \epsilon_n \in F_n$ denote a fixed choice of

generators of F_n and define the subset of F_n consisting of the lexicographically ordered words of length ≤ 3 with no repeated letters

$$H_n = \{\epsilon_j,\ 1 \leq j \leq n\} \cup \{\epsilon_j\epsilon_k,\ 1 \leq j < k \leq n\} \cup \{\epsilon_j\epsilon_k\epsilon_l,\ 1 \leq j < k < l \leq n\} \subset F_n,$$

of cardinality $N = n + \binom{n}{2} + \binom{n}{3} = \frac{n^3+5n}{6}$. One can show that given any word $w \in F_n$, the function t_w is a polynomial with rational coefficients in the variables t_γ, $\gamma \in H_n$ (see [V], Cor. 4.14). These generators give an embedding of the categorical quotient in $\mathbb{C}^N$, so that $X_n /\!\!/ G$ corresponds to some polynomial ideal in $\mathbb{C}[t_\gamma,\ \gamma \in H_n]$.

In the present paper, inspired by geometric invariant theory (GIT), we obtain a simple criterion, in terms of trace functions (1.1), for an element $A \in X_n$ to be in the subset X_n^{st} of $X_n = G^{\times n}$ of stable points for the action of G. By standard arguments of GIT, the *affine stable quotient* X_n^{st}/G will be an affine variety which is a geometric quotient of X_n^{st} by G in the sense that all fibers of the quotient map are indeed orbits of the action. This is in contrast to the categorical affine quotient $X_n /\!\!/ G$ where the points only parametrize closure-equivalence classes of orbits.

In another direction, we show that a similar numerical criterion can be used to check the irreducibility of a representation of a finitely generated group Γ in $G = SL(2,\mathbb{C})$, as follows. Let $\bar{\epsilon}_1, ..., \bar{\epsilon}_n$ be a choice of generators of Γ. If $\rho : \Gamma \to G$ is a representation, define $A \in X_n$ by setting

$$A = (\rho(\bar{\epsilon}_1), ..., \rho(\bar{\epsilon}_n)).$$

On the other hand, a given point $A = (A_1, ..., A_n) \in X_n$ produces a representation ρ of the free group F_n, by letting $\rho(\epsilon_j) = A_j$ for $j = 1, ..., n$. For three indices $1 \leq j, k, l \leq n$, denote by $\rho_{jkl} : F_3 \to G$ the representation associated to $(\rho(\bar{\epsilon}_j), \rho(\bar{\epsilon}_k), \rho(\bar{\epsilon}_l)) \in X_3$. In section 4, we show.

Theorem 1.1. *For a representation $\rho : \Gamma \to G$, the following sentences are equivalent.*
(i) $\rho : \Gamma \to G$ is reducible.
(ii) $(\rho(\bar{\epsilon}_1), ..., \rho(\bar{\epsilon}_n)) \in X_n$ is not stable for the conjugation action of G.
(iii) There is a $g \in SL(2,\mathbb{C})$ such that all matrices $g\rho(\bar{\epsilon}_k)g^{-1}$ are upper triangular.
(iv) For all triples of indices $1 \leq j, k, l \leq n$, $\rho_{jkl} : F_3 \to G$ is reducible.

In [CS] (Cor. 1.2.2), Culler and Shalen proved that $\rho : \Gamma \to G$ is reducible if and only if $\mathsf{tr}(h) = 2$ for every element h in the commutator subgroup $[\Gamma, \Gamma]$. Part (iv) of Theorem 1.1 shows that irreducibility can be decided by a finite process, looking only at all the associated representations of F_3. Furthermore, the analysis of irreducibility for the case of F_3 leads to the following concrete numerical condition. For each triple of indices $1 \leq j, k, l \leq n$, define the following G invariant functions $\sigma_{jk}, \Delta_{jkl} : X_n \to \mathbb{C}$ (see also definition 2.4 below).

$$\begin{aligned} \sigma_{jk}(A) &= \mathsf{tr}([A_j, A_k]) - 2, \\ \Delta_{jkl}(A) &= (\mathsf{tr}(A_jA_kA_l) - \mathsf{tr}(A_lA_kA_j))^2, \end{aligned} \tag{1.2}$$

where $[A_j, A_k] = A_jA_kA_j^{-1}A_k^{-1}$ is the commutator of $SL(2,\mathbb{C})$ matrices. The function Δ_{jkl} may be called the *Fricke discriminant*, being the discriminant of the polynomial associated to the Fricke relation (see [Go, M]). We prove

Theorem 1.2. *Let $A = (A_1, ..., A_n) \in X_n$ be the n-tuple associated with the representation $\rho : \Gamma \to G$. Then ρ is reducible if and only if $\sigma_{jk}(A) = \Delta_{jkl}(A) = 0$ for any triple $1 \leq j, k, l \leq n$.*

The computations involved in the theorems above can be easily adapted for the case of $SL(2,\mathbb{C})$ acting by simultaneous conjugation on the vector space V_n of n-tuples of arbitrary complex 2×2 matrices. In section 3, after briefly recalling the relevant definitions in geometric invariant theory, we describe the stable locus for this bigger space, and prove results analogous to the above theorem (see Theorem 3.3). All these results are based on the explicit characterization of the n-tuples of 2×2 matrices that are simultaneously similar to a set of n upper triangular matrices, in terms of invariant functions, which is obtained in section 2 (see Theorem 2.7). In section 4, we also briefly comment on the relation between this notion of stability and the stability of the holomorphic vector bundle on a compact Riemann surface S arising from a representation of the fundamental group of S into G (see Proposition 4.4).

Section 5 focus on the problem of reconstructing an orbit of the action of G on X_n from a minimal number of traces, which was motivated by the articles [M] and [Go]. Consider an arbitrary finite sequence of words $J = (w_1, ..., w_N) \in (F_n)^N$ and let T_J denote the map

$$\begin{aligned} T_J : X_n \cong Hom(F_n, G) &\rightarrow \mathbb{C}^N \\ \rho &\mapsto (t_{w_1}(\rho), ..., t_{w_N}(\rho)). \end{aligned} \tag{1.3}$$

Given that the quotient $G^{\times n}/\!\!/G$ is a variety of dimension $3n-3$, and T_J factors through this quotient, it is natural to look for a sequence J of $N = 3n-3$ words, such that T_J is surjective onto a Zariski open subset of $\mathbb{C}^{3n-3}$ and that all the preimages are finite (when non empty). Under the algebra-geometry dictionary, this is equivalent to finding a minimal set of trace functions $t_{w_1}, ..., t_{w_N}$ such that the field of invariant rational functions on $G^{\times n}$ is an algebraic extension of $\mathbb{C}(t_{w_1}, ..., t_{w_N})$.

In this paper, we consider only those sequences J composed of the n basic trace functions of length one $t_{\epsilon_1}, ..., t_{\epsilon_n}$ and some choice of $2n-3$ other words of length 2. The basic example is the result attributed to Vogt and Fricke that, for $n = 2$, the map

$$T_{(\epsilon_1, \epsilon_2, \epsilon_1\epsilon_2)} : X_2 \rightarrow \mathbb{C}^3$$

is surjective. In [Go], Goldman presents an almost self-contained proof of this, showing also that for $n = 3$, the trace map

$$T_{(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_1\epsilon_2, \epsilon_1\epsilon_3, \epsilon_2\epsilon_3)} : X_3 \rightarrow \mathbb{C}^6$$

is again surjective. To consider the case of a free group of arbitrary rank $n \geq 4$, first note that we cannot take as J the sequence with the n words of length one together with the $\binom{n}{2}$ ordered words of length two, since this would have length greater than the wanted $3n-3$. Let us choose the sequence of length $3n-3$ omitting those words of length two $\epsilon_j\epsilon_k$ such that $\{j, k\} \cap \{1, 2\} = \emptyset$,

$$J_n := (\epsilon_1, \epsilon_2, \epsilon_1\epsilon_2, ..., \epsilon_k, \epsilon_1\epsilon_k, \epsilon_2\epsilon_k, ..., \epsilon_n, \epsilon_1\epsilon_n, \epsilon_2\epsilon_n), \tag{1.4}$$

and denote the corresponding trace map by $T_n = T_{J_n}$. Of course, it would be completely analogous to use another sequence of the form (1.4) with another pair of indices playing the role of $\{1, 2\}$. In terms of the n-tuple $A = (A_1, ..., A_n) \in G^{\times n}$ corresponding to $\rho : F_n \rightarrow G$ we have

$$T_n(A) = (t_1, t_2, t_{12}, ..., t_k, t_{1k}, t_{2k}, ..., t_n, t_{1n}, t_{2n}), \tag{1.5}$$

where we use the notation $t_j = \mathsf{tr}(A_j)$, $t_{jk} = \mathsf{tr}(A_jA_k)$. In section 5, we show that T_n is almost surjective for $n \geq 4$, and omits a set contained in a very explicit irreducible subvariety of $\mathbb{C}^{3n-3}$ of codimension 1. Examples of points $\mathbf{z} \in \mathbb{C}^{3n-3}$, with $T_n^{-1}(\mathbf{z})$ empty are given in the Appendix.

The algebraic map $T_n : X_n \rightarrow \mathbb{C}^{3n-3}$ (1.5) will be called the *Magnus trace map*. In [M], Magnus showed that, given three matrices A_1, A_2 and A_3 verifying $\sigma_{12}(A_1, A_2) \neq 0$ and $\Delta_{123}(A_1, A_2, A_3) \neq 0$ and given any $\mathbf{q} \in \mathbb{C}^{3n-9}$ (thought as the last $3n-9$ coordinates in $\mathbb{C}^{3n-3}$) one can find $n-3$ other matrices $A_4, ..., A_n$ such that $T_n(A) = (\mathbf{p}, \mathbf{q})$, where $\mathbf{p} = (t_1, t_2, t_{12}, t_3, t_{13}, t_{23})$. He also proved that the number of different solutions $(A_4, ..., A_n)$ realizing this equation is bounded by 2^{n-2}. It turns out that the Fricke discriminant condition $\Delta_{123} \neq 0$ is not really necessary, and we only need to impose the condition $\sigma_{12}(A_1, A_2) \neq 0$ to be able to find $A_3, ..., A_n$ such that $\mathsf{tr}(A_k)$, $\mathsf{tr}(A_1A_k)$ and $\mathsf{tr}(A_2A_k)$ assume preassigned values for $k = 3, ..., n$. Moreover, the condition $\sigma_{12}(A_1, A_2) \neq 0$ allows us to express the (at most 2^{n-2}) orbits in the preimage of T_n in very explicit terms.

Theorem 1.3. *Let $B_1, B_2 \in SL(2,\mathbb{C})$ be such that $\mathsf{tr}([B_1, B_2]) \neq 2$. Then, there exists a $g \in G$ such that $A_j := gB_jg^{-1}$, $j = 1, 2$ are invariant under transposition. Let $n \geq 3$ and $\mathbf{r} = (t_1, t_2, t_{12})$. Then, given any $\mathbf{s} \in \mathbb{C}^{3n-6}$ there exist $n-2$ matrices $A_3, ..., A_n \in SL(2,\mathbb{C})$ such that*

$$T_n(A) = (\mathbf{r}, \mathbf{s}). \tag{1.6}$$

Moreover, given any solution $A \in X_n$ of (1.6) with $\mathsf{tr}([A_1, A_2]) \neq 2$ and $A_1^T = A_1$ and $A_2^T = A_2$ (where T denotes transposition) the inverse image $T_n^{-1}(\mathbf{r}, \mathbf{s})$ consists of the G orbits of the finite set

$$\left\{(A_1, A_2, B_3, ..., B_n) : B_j = A_j \text{ or } B_j = A_j^T \text{ for } j = 3, ..., n\right\}.$$

We would like to mention that most of the methods in this article are constructive, in the sense that they can be used to implement algorithms to effectively compute the quantities involved.

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2. Degenerate simultaneous similarity of 2×2 matrices

In this section, we are interested in the simultaneous conjugacy classes of a finite set of 2×2 complex matrices. We will describe the most degenerate cases, in particular give necessary and sufficient conditions, in terms of invariant functions, for n matrices to be simultaneously conjugated to matrices in upper or lower triangular form.

Let the general linear group $GL(2,\mathbb{C})$ act on the vector space of n-tuples of 2×2 matrices ($n \geq 1$)

$$V_n := (M_{2\times 2}(\mathbb{C}))^{\times n}$$

by simultaneous conjugation

$$g \cdot A := (gA_1g^{-1}, ..., gA_ng^{-1}), \tag{2.1}$$

where $A = (A_1, ..., A_n) \in V_n$ and $g \in GL(2,\mathbb{C})$. There are plenty of $GL(2,\mathbb{C})$-invariant regular (i.e, polynomial) functions on V_n and, by the first fundamental theorem of invariants of $m \times m$ matrices [P], the trace functions defined by

$$V_n \to \mathbb{C}, \qquad A \mapsto \mathsf{tr}(A_{i_1} \cdots A_{i_k})$$

and labelled by 'words' $A_{i_1} \cdots A_{i_k}$ in the components A_j of $A \in V_n$, generate the ring of invariants $\mathbb{C}[V_n]^{GL(2,\mathbb{C})}$. Moreover, this ring is finitely generated and we have

Theorem 2.1. (Procesi [P]) *Any $GL(2,\mathbb{C})$-invariant regular function on V_n is a polynomial in the following set of $(n+1)^3$ functions*

$$A \mapsto \mathsf{tr}(A_jA_kA_l), \qquad 0 \leq j,k,l \leq n,$$

where $A = (A_1, ..., A_n) \in V_n$ and $A_0 = I$ is the identity 2×2 matrix.

The many relations between these functions, described by the second fundamental theorem of invariants of matrices (see [P]), will not be important here.

As in the case $n = 1$, two elements A and A' of V_n will be called *similar* if they are in the same $GL(2,\mathbb{C})$ orbit. Note that an element $A \in V_n$ can be viewed either as a vector of 2×2 matrices as above or, alternatively, as a *single* matrix with vector valued entries

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}^n. \tag{2.2}$$

This justifies the following terminology and notation.

Definition 2.2. A point $A = (A_1, ..., A_n) \in V_n$ will be called an n-matrix. We will say that A is an *upper triangular n-matrix* if the vector $c \in \mathbb{C}^n$ is zero, and we will denote by $\mathcal{UT}_n \subset V_n$ the $GL(2,\mathbb{C})$ orbit of the subset of upper triangular n-matrices. Hence, $A \in \mathcal{UT}_n$ if and only if A is *similar* to an upper triangular n-matrix. A_j is called the jth *component* of A.

Note that A is similar to an upper triangular n-matrix if and only if there is a proper nonzero subspace of $\mathbb{C}^2$ which is fixed by every component A_j of A. The condition $A \in \mathcal{UT}_n$ is also equivalent to A being similar to a lower triangular n-matrix (one with zero $b \in \mathbb{C}^n$).

The similarity classes of a pair of $m \times m$ matrices were obtained in [Fr], and in the simplest $m = 2$, $n = 2$ case, the following irreducible algebraic subset of V_2 plays an important role

$$W = \left\{ (A_1, A_2) \in V_2 : (t_{11} - \frac{1}{2}t_1^2)(t_{22} - \frac{1}{2}t_2^2) = \left(t_{12} - \frac{1}{2}t_1t_2\right)^2 \right\}. \tag{2.3}$$

Here and below, we are using the following notation

$$\begin{aligned} t_j &:= \mathsf{tr}(A_j) \\ t_{jk} &:= \mathsf{tr}(A_j A_k) \end{aligned}$$

for a general element $A = (A_1, ..., A_n) \in V_n$ and any pair of indices $j, k \in \{1, ..., n\}$. The relevance of W can be seen from the fact that if (A_1, A_2) does not belong to W, then its $GL(2,\mathbb{C})$ orbit is uniquely determined by $t_1, t_2, t_{11}, t_{22}, t_{12}$ ([Fr]). Moreover, the following is not difficult to prove.

Proposition 2.3. (see, for instance, [Fr]) *A pair $(A_1, A_2) \in V_2$ belongs to W if and only if it is in the orbit of a pair of upper triangular matrices.*

To generalize this result to higher n, let us abbreviate some frequently used $GL(2,\mathbb{C})$-invariant functions on V_n as follows

Definition 2.4. Define, for every triple of indices $1 \leq j, k, l \leq n$,

$$\begin{aligned} \tau_{jk} &:= t_{jk} - \frac{1}{2} t_j t_k, \\ \sigma_{jk} &:= \tau_{jk}^2 - \tau_{jj}\tau_{kk}, \\ \Delta_{jkl} &:= (t_{jkl} - t_{lkj})^2. \end{aligned}$$

We omit the dependence on $A = (A_1, ..., A_n)$ where no ambiguity arises. For $A \in V_n$ and $g \in GL(2,\mathbb{C})$, we will always write

$$A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \qquad j = 1, ..., n, \qquad g = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

or sometimes $A_j = (a_j, b_j, c_j, d_j)$, $g = (x, y, z, w)$. In terms of these variables and of $e_j := a_j - d_j$, we have:

$$\begin{aligned} \tau_{jk} &= \frac{e_j e_k}{2} + b_j c_k + c_j b_k, \\ \sigma_{jk} &= (b_j c_k - c_j b_k)^2 - (b_j e_k - e_j b_k)(c_j e_k - e_j c_k). \end{aligned} \tag{2.4}$$

We also use the abbreviation

$$\nu_j := \tau_{jj} = t_{jj} - \frac{1}{2} t_j^2 = \frac{e_j^2}{2} + 2 b_j c_j,$$

so that $\nu_j \neq 0$ if and only if A_j has distinct eigenvalues. The functions σ_{jk} and Δ_{jkl} are fully symmetric under index permutation and vanish whenever two indices coincide. Note that $-\sigma_{jk}$ and $-\frac{1}{2}\Delta_{jkl}$ are, respectively, the top left 2×2 minor and the determinant of the symmetric matrix

$$\begin{pmatrix} \tau_{jj} & \tau_{jk} & \tau_{jl} \\ \tau_{kj} & \tau_{kk} & \tau_{kl} \\ \tau_{lj} & \tau_{lk} & \tau_{ll} \end{pmatrix}. \tag{2.5}$$

With slightly different normalizations, the restrictions of these functions to $SL(2,\mathbb{C})^{\times n}$ were used in [GM] and [M]. Since the equation (2.3) that defines $W \subset V_2$ is $\sigma_{12} = 0$, the condition $\sigma_{jk}(A) = 0$ for some $j, k \in \{1, ..., n\}$ is equivalent to $(A_j, A_k) \in \mathcal{UT}_2$, by Proposition 2.3. Therefore, we have

Proposition 2.5. *If $A \in V_n$ is similar to an upper triangular n-matrix, then $\sigma_{jk}(A) = 0$ for all distinct $1 \leq j, k \leq n$.*

Proof. If $A = (A_1, ..., A_n) \in \mathcal{UT}_n$ then $g \cdot A$ is upper triangular, for some $g \in G$. Hence, $g \cdot (A_j, A_k)$ is an upper triangular 2-matrix, for any $1 \leq j, k \leq n$, and $\sigma_{jk} = 0$ by Proposition 2.3. $\square$

For $n \geq 2$, define, for distinct $j, k \in \{1, ..., n\}$, the following algebraic subsets of V_n

$$W_{jk} = W_{kj} := \{A \in V_n : \sigma_{jk}(A) = 0\},$$

$$\Sigma_n := \bigcap_{j,k} W_{jk}.$$

Then, $A \in \Sigma_n$ if and only if every pair of matrix components of A is in $\mathcal{UT}_2$. From Proposition 2.5, we have

$$\mathcal{UT}_n \subset \Sigma_n \subset V_n$$

for all $n \geq 2$. However, the vanishing of all σ_{jk} is not sufficient for A to be in $\mathcal{UT}_n$, for $n \geq 3$, as the next example shows.

Example 2.6. Let $A = (A_1, A_2, A_3)$ be defined by

$$A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_3 & 0 \\ c_3 & d_3 \end{pmatrix}.$$

Then $\sigma_{12} = \sigma_{13} = 0$ and $\sigma_{23} = b_2c_3(e_2e_3 + b_2c_3)$. Assume that $e_2e_3 + b_2c_3 = 0$ and that $e_1b_2c_3 \neq 0$, so that all σ_{jk} vanish, neither A_2 or A_3 are diagonal, and (since these assumptions imply $e_2e_3 \neq 0$) all three matrices have distinct eigenvalues. Let $g = (x, y, z, w) \in SL(2,\mathbb{C})$. Then

$$\begin{aligned} gA_1g^{-1} &= \begin{pmatrix} * & -xye_1 \\ zwe_1 & * \end{pmatrix} \\ gA_2g^{-1} &= \begin{pmatrix} * & x(xb_2 - ye_2) \\ z(we_2 - zb_2) & * \end{pmatrix} \\ gA_3g^{-1} &= \begin{pmatrix} * & -y(xe_3 + yc_3) \\ w(ze_3 + wc_3) & * \end{pmatrix}, \end{aligned} \tag{2.6}$$

from which it follows that A is not similar to an upper triangular 3-matrix.

On the other hand, we have.

Theorem 2.7. *Let $n \geq 1$. $A = (A_1, ..., A_n) \in V_n$ is similar to an upper triangular matrix if and only if for all triples $1 \leq j, k, l \leq n$, (A_j, A_k, A_l) is similar to an upper triangular 3-matrix.*

For the proof, we will use the following lemmata.

Lemma 2.8. *Let A be an upper triangular n-matrix with $\nu_j = \frac{e_j^2}{2} \neq 0$, for some $1 \leq j \leq n$. Then, A is similar to another upper triangular n-matrix $A' = (A'_1, ..., A'_n)$ with A'_j diagonal.*

Proof. We only need to find $g \in GL(2,\mathbb{C})$ such that gA_kg^{-1} is still upper triangular for any k, and such that gA_jg^{-1} is diagonal. Letting $g = (x, y, 0, x^{-1})$ for some $x \neq 0$, we calculate $gA_kg^{-1} = (a_k, b_kx^2 - yxe_k, 0, d_k)$, for every $k = 1, ..., n$. Therefore, all gA_kg^{-1} are upper triangular matrices, and using $y = b_jx/e_j$ (since $e_j \neq 0$), gA_jg^{-1} is diagonal. □

Lemma 2.9. *As in Example 2.6, let $A = (A_1, A_2, A_3)$ be a triple of the form*

$$A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_3 & 0 \\ c_3 & d_3 \end{pmatrix}, \tag{2.7}$$

Then $A \in \mathcal{UT}_3$ if and only if $e_1b_2c_3 = 0$.

Proof. If $e_1b_2c_3 = 0$ at least one of the factors is zero. In each case, A_1 is a scalar, A is lower triangular, or A is upper triangular, respectively, so $A \in \mathcal{UT}_3$. Conversely, suppose that $e_1b_2c_3 \neq 0$. Then, from equations (2.6) we see that there is no $g \in G$ that will make $g \cdot A$ upper or lower triangular, so $A \notin \mathcal{UT}_3$. □

We can now finish the proof of Theorem 2.7.

Proof. The Theorem is obvious for $n \leq 3$, so let $n \geq 4$. If $A \in V_n$ is similar to an upper triangular n-matrix, then obviously any m-tuple $(A_{j_1}, ..., A_{j_m})$ of $m \leq n$ components of A will be in $\mathcal{UT}_m$. Conversely, let all triples of components be in $\mathcal{UT}_3$ and suppose, by induction, that the result is valid for $n-1$. Then, in particular, all the σ_{jk} and all Δ_{jkl} are zero, for indices j, k, l between 1 and $n-1$. To reach a contradiction, assume that A is not similar to an upper triangular n-matrix. By induction, we can suppose that $(A_1, ..., A_{n-1})$ has been conjugated so that it is already an upper triangular $(n-1)$-matrix. Let $A_n = (a_n, b_n, c_n, d_n)$ with $c_n \neq 0$. None of the A_j can be central,

otherwise the result would follow by induction. The $n-1$ conditions $\sigma_{jn} = 0$, $j = 1, ..., n-1$ imply (because $c_n \neq 0$)

$$(2.8) \qquad b_j^2 c_n + b_j e_j e_n - e_j^2 b_n \;=\; 0, \quad \text{for } j = 1, ..., n-1.$$

If one of the e_j, $j = 1, ..., n-1$ vanishes then, since A_j is non-scalar, $b_j \neq 0$ and the equations (2.8) become $b_j^2 c_n = 0$ and have no solution. As a consequence, none of these e_j's can be zero. Then, by lemma 2.8, one can assume that $b_1 = 0$, and the equation (2.8) with $j = 1$ implies $b_n = 0$. Now, we have all triples (A_1, A_k, A_n), with $k = 2, ..., n-1$, in the form (2.7). Since $(A_1, A_k, A_n) \in \mathcal{UT}_3$ by hypothesis, lemma 2.9 implies $e_1 b_k c_n = 0$. So, $b_k = 0$ for all $k = 2, ..., n-1$. But then $(A_1, ..., A_n)$ is lower triangular, and we have a contradiction. □

Note that the Theorem is true for any algebraically closed field of characteristic 0 in place of the field of complex numbers. The following statement is also useful.

Lemma 2.10. *Let $A = (A_1, A_2, A_3) \in \Sigma_3 \setminus \mathcal{UT}_3$. Then, A is similar to a triple of the form (2.7) with $e_2 e_3 + b_2 c_3 = 0$ and $e_1 b_2 c_3 \neq 0$.*

Proof. Assume that $A \in \Sigma_3$. In particular, $\sigma_{12} = 0$, so we can suppose that A_1 and A_2 are both upper triangular. Let $A_3 = (a_3, b_3, c_3, d_3)$. Since $A \notin \mathcal{UT}_3$, none of the A_j can be a scalar, and c_3 is nonzero. The 2 conditions $\sigma_{j3} = 0$, $j = 1, 2$ imply (because $c_3 \neq 0$)

$$(2.9) \qquad b_j^2 c_3 + b_j e_j e_3 - e_j^2 b_3 \;=\; 0, \quad \text{for } j = 1, 2.$$

If one of the e_j, $j = 1, 2$ vanishes then, since A_j is non-scalar, $b_j \neq 0$ and the equations (2.9) become $b_j^2 c_3 = 0$ and have no solution. So, necessarily $e_1 e_2 \neq 0$. Then, by lemma 2.8, one can assume that $b_1 = 0$, and the equation (2.9) with $j = 1$ implies $b_3 = 0$. Since A cannot be lower triangular, we need to have $b_2 \neq 0$, and (2.9) for $j = 2$ simplifies to $b_2 c_3 + e_2 e_3 = 0$. □

Corollary 2.11. *$(A_1, A_2, A_3) \in V_3$ is similar to an upper triangular 3-matrix if and only if $\sigma_{12} = \sigma_{13} = \sigma_{23} = \Delta_{123} = 0$.*

Proof. If $A \in \mathcal{UT}_3$, then $\sigma_{jk} = 0$ for all j, k, by Proposition 2.5 and by direct computation $t_{123} = t_{321}$. Conversely, if all $\sigma_{jk} = 0$ and $A \notin \mathcal{UT}_3$, by lemma 2.10, we can suppose that A is in the form (2.7) with $e_1 b_2 c_3 \neq 0$. An easy calculation then gives $\Delta_{123} = (t_{123} - t_{321})^2 = e_1^2 b_2^2 c_3^2 \neq 0$. □

This Corollary extends to all 2×2 matrices the result stated in Prop. 4.4 of [GM] for triples of $SL(2,\mathbb{C})$ matrices. The following Proposition will be useful later.

Proposition 2.12. *Let A be a 2-matrix with $\sigma_{12} \neq 0$. Then A is similar to a 2-matrix B invariant under simultaneous transposition ($B_1^T = B_1$ and $B_2^T = B_2$).*

Proof. We consider first the case when at least one of the matrices A_1 or A_2 is diagonalizable. Without loss of generality let A_1 be diagonal and $A_2 = (a_2, b_2, c_2, d_2)$. Then $\sigma_{12} = -e_1^2 b_2 c_2$ from (2.4). Conjugation of A by the diagonal matrix $g = (x, 0, 0, x^{-1})$ produces the assignment $b_2 \mapsto b_2 x^2$, $c_2 \mapsto c_2 x^{-2}$ and so, we just solve $b_2 x^2 = c_2 x^{-2}$ for an appropriate $x \neq 0$, which is possible since $\sigma_{12} \neq 0$. Now, consider the case $\nu_1 = \nu_2 = 0$. Assuming that A_1 is already in Jordan canonical form, write $A = (A_1, A_2)$ as

$$(2.10) \qquad A_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

with $b_1 c_2 \neq 0$ according to the hypothesis $\sigma_{12} \neq 0$. Suppose first that $b_2 = 0$, which implies $d_2 = a_2$, since $\nu_2 = 0$. Conjugating (A_1, A_2) by a diagonal matrix as before, we can further assume that $b_1 = c_2$. Then, using

$$g = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

and we obtain explicitly the transposition invariant pair,

$$B_1 = g \cdot A_1 = \begin{pmatrix} a_1 + \lambda & i\lambda \\ i\lambda & a_1 - \lambda \end{pmatrix}, \qquad B_2 = g \cdot A_2 = \begin{pmatrix} a_2 - \lambda & i\lambda \\ i\lambda & a_2 + \lambda \end{pmatrix},$$

with $b_1 = c_2 = 2i\lambda$. Finally, if $b_2 \neq 0$ in (2.10), using the equation $\nu_2 = \frac{e_2^2}{2} + 2b_2c_2 = 0$, which implies $e_2 \neq 0$, it is a simple computation to show that conjugation by $g = (x, 1, 0, x^{-1})$ with $x = \frac{e_2}{2b_2}$ reduces that pair to one with A_1 upper and A_2 lower triangular. Note that in both cases, the transposition invariant 2-matrix B verifies $\sigma_{12} = -e_1^2 b_2^2 = 16\lambda^4$. □

3. Stability of the $SL(2,\mathbb{C})$ action on 2×2 matrices

In this section, we determine, in terms of invariant functions, the stable points of the action of $SL(2,\mathbb{C})$ on the vector space V_n of n-tuples of complex 2×2-matrices, under simultaneous conjugation on each factor.

Recall that, in the general situation of a general algebraic linearly reductive Lie group K acting on an affine variety V one defines the affine quotient variety $V/\!\!/K$ as the spectrum of the ring of invariant functions on V, which comes equipped with a projection

$$q : V \to V/\!\!/K$$

induced from the canonical inclusion of algebras $\mathbb{C}[V] \subset \mathbb{C}[V]^K$. The set of closed orbits is in bijective correspondence with geometric points of the quotient $V/\!\!/K$. Recall also that a vector $x \in V$ is said to be *stable* if the corresponding map

$$\psi_x : K \to V, \qquad g \mapsto g \cdot x$$

is proper. It is easy to see that $x \in V$ is stable if and only if the closure of the K-orbit of x does not intersect the closed subset consisting of points $x \in V$ with positive dimensional stabilizer subgroup. Another useful criterion for stability is the *Hilbert-Mumford numerical criterion*, which is stated in terms of nontrivial homomorphisms $\phi : \mathbb{C}^* \to K$, called one parameter subgroups (1PS) of K. To any such ϕ and to a point $x \in V$ one associates the morphism $\phi_x : \mathbb{C}^* \to V$ given by mapping $\lambda \in \mathbb{C}^*$ to the point $\phi(\lambda)\cdot x$. If ϕ_x can be extended to a morphism $\overline{\phi_x} : \mathbb{C} \to V$, we say that $\lim_{\lambda\to 0} \phi_x$ exists and equals $\overline{\phi_x}(0)$.

Theorem 3.1. (Hilbert-Mumford [MFK], see also [Gi]) *A point $x \in V$ is stable if and only for every one parameter subgroup ϕ of K, ϕ_x cannot be extended to a morphism $\mathbb{C} \to V$.*

It is easy to see that the conjugation action of $GL(2,\mathbb{C})$ on the space of n-tuples of 2×2 matrices V_n has no stable points, since the scalar nonzero matrices will stabilize any point $A \in V_n$. This is not a big problem, since the same orbit space can be obtained with the conjugation action of $G \equiv SL(2,\mathbb{C})$ on V_n which has generically finite stabilizers. This is just the restriction to $G \subset GL(2,\mathbb{C})$ of the action (2.1). One could as well consider the action of $PSL(2,\mathbb{C})$ which would have generically trivial stabilizers, but we will keep using $G = SL(2,\mathbb{C})$. It is clear that any diagonal n-matrix A (one for which both vectors b and c in (2.2) are zero) has the subgroup

$$H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \subset G,\ \lambda \in \mathbb{C}^* \tag{3.1}$$

contained in its stabilizer. The following is then an easy application of the Hilbert-Mumford criterion, Theorem 3.1.

Proposition 3.2. *$A \in V_n$ is stable if and only if A is not similar to an upper triangular n-matrix.*

Proof. If A is an upper triangular n-matrix, a simple computation shows that the closure of the orbit of A under the subgroup $H \subset G$ (3.1) will intersect D. Therefore, no point in the orbit of A will be stable. Conversely, let $A \in V_n$ be not stable and apply the numerical criterion. By elementary representation theory, any one parameter subgroup of G is conjugated to

$$\lambda \mapsto \phi_n(\lambda) = \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}, \quad n \in \mathbb{N}_0.$$

In other words, any 1PS can be written as $\phi = g^{-1}\phi_n g$, for some $g \in G$ and some ϕ_n so,

$$\lim_{\lambda\to 0} \phi(\lambda)\cdot A = g^{-1} \lim_{\lambda\to 0} \phi_{\mathsf{n}}(\lambda)\cdot (g\cdot A). \tag{3.2}$$

Writing $g \cdot A$ as

$$g \cdot A = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix},$$

we obtain

$$\phi_n(\lambda) \cdot (g \cdot A) = \begin{pmatrix} a(g) & b(g)\lambda^{2n} \\ c(g)\lambda^{-2n} & d(g) \end{pmatrix}.$$

By the Hilbert-Mumford criterion, the limit (3.2) exists for some 1PS, so we must have $c(g) = 0$, for some $g \in G$. This means that $g \cdot A$ is an upper triangular n-matrix. □

Note that this result can be easily generalized to describe the stable points of the action of $SL(m,\mathbb{C})$ under simultaneous conjugation on the vector space of n-tuples of $m \times m$ matrices. To summarize, for the action of $SL(2,\mathbb{C})$ on V_n we have shown the following, which is a consequence of Theorem 2.7 and Proposition 3.2.

Theorem 3.3. *The following are equivalent for an element $A = (A_1, ..., A_n) \in V_n$, $n \geq 1$.*
(i) $A \in V_n$ is stable.
(ii) There exists $1 \leq j, k, l \leq n$ such that $(A_j, A_k, A_l) \in V_3$ is stable.
(iii) There exists $1 \leq j, k, l \leq n$ such that $\sigma_{jk}(A) \neq 0$ or $\Delta_{jkl}(A) \neq 0$.
(iv) A is not similar to an upper triangular n-matrix.
(v) There is no proper nonzero subspace of $\mathbb{C}^2$ preserved by the set $\{A_1, ..., A_n\}$. □

4. Irreducibility of representations of finitely generated groups

We now use the numerical condition for stability found above to derive a similar criterion for irreducibility of a representation of a finitely generated group Γ into $G = SL(2,\mathbb{C})$. As in the introduction, by fixing a set of generators $\bar{\epsilon}_1, ..., \bar{\epsilon}_n$ of Γ, we associate to a representation $\rho : \Gamma \to G$ the point $A = (A_1, ..., A_n) \in X_n$ given by

$$A = (\rho(\bar{\epsilon}_1), ..., \rho(\bar{\epsilon}_n)).$$

In this section we are therefore dealing with the conjugation action of $G = SL(2,\mathbb{C})$ restricted to the affine subvariety $X_n = G^{\times n} \subset V_n$. For G-invariant functions on X_n, we continue to use the same notations described in Definition 2.4, in particular, we still denote by t_j (resp. t_{jk}) trace of the matrix A_j (resp. $A_j A_k$). Because of the standard identities

$$\begin{aligned} \mathsf{tr}(B_1B_2) + \mathsf{tr}(B_1^{-1}B_2) &= \mathsf{tr}(B_1)\mathsf{tr}(B_2) \\ \mathsf{tr}(B_1) &= \mathsf{tr}(B_1^{-1}) \\ \mathsf{tr}(B_1^2) &= \mathsf{tr}^2(B_1) - 2, \end{aligned}$$

valid for any two $SL(2,\mathbb{C})$ matrices B_1, B_2, some of those G invariant functions acquire a new form

$$\begin{aligned} \nu_j &= \frac{t_j^2}{2} - 2, \\ \sigma_{jk} &= t_j^2 + t_k^2 + t_{jk}^2 - t_j t_k t_{jk} - 4 = \mathsf{tr}([A_j, A_k]) - 2, \end{aligned}$$

where $[B_1, B_2] = B_1B_2B_1^{-1}B_2^{-1}$ denotes the commutator of two $SL(2,\mathbb{C})$ matrices.

Recall that a representation $\rho : \Gamma \to G$ is called irreducible if there are no proper nonzero subspaces of $\mathbb{C}^2$ which are invariant under $\rho(\Gamma)$. Therefore, if all matrices $\rho(\gamma)$, $\gamma \in \Gamma$ are upper triangular, ρ is reducible. In this case, $\mathsf{tr}([\rho(h_1), \rho(h_2)]) = 2$ for any $h_1, h_2 \in \Gamma$. This condition is also sufficient for reducibility as proved in [CS].

Theorem 4.1. (Culler & Shalen [CS]) *A representation $\rho : \Gamma \to G$ is irreducible if and only if $\mathsf{tr}(c) \neq 2$ for some element c of the commutator subgroup $[\Gamma, \Gamma]$.*

Making use of Theorem 3.3, we obtain necessary and sufficient conditions for irreducibility, depending only on the n-matrix A associated to ρ, via the fixed choice of generators of Γ.

Theorem 4.2. *Let $A = (\rho(\bar{\epsilon}_1), ..., \rho(\bar{\epsilon}_n)) \in X_n$ be the n-tuple associated with the representation $\rho : \Gamma \to G$. Then, the following sentences are equivalent.*

(i) $A \in X_n$ is stable.

(ii) There exists $1 \leq j, k, l \leq n$ such that $(\rho(\bar{\epsilon}_j), \rho(\bar{\epsilon}_k), \rho(\bar{\epsilon}_l)) \in X_3$ is stable.

(iii) There exists $1 \leq j, k, l \leq n$ such that $\sigma_{jk}(A) \neq 0$ or $\Delta_{jkl}(A) \neq 0$.

(iv) A is not similar to an upper triangular n-matrix.

(v) $\rho : \Gamma \to G$ is irreducible.

Proof. The equivalence between (i)-(iv) follows easily from the equivalence of (i)-(iv) for the $SL(2,\mathbb{C})$ action on V_n proved in Theorem 3.3. Let us show that (iv) is equivalent to (v). If $A \notin \mathcal{UT}_n$ there is no subspace of $\mathbb{C}^2$ preserved by the set $\{\rho(\bar{\epsilon}_1), ..., \rho(\bar{\epsilon}_n)\}$, so ρ is irreducible. Conversely, if $A \in \mathcal{UT}_n$ then, after conjugating ρ with some $g \in G$, all the matrices $\rho(h)$, $h \in \Gamma$ will be upper triangular because the $\bar{\epsilon}_j$ are the generators of Γ. So ρ is reducible. □

This result completes the proof of theorems 1.1 and 1.2; it can be viewed as a sharpening of Theorem 4.1 (it also generalizes prop. 1.5.5 of [CS]), since part (iii) implies that irreducibility of a representation $\rho : \Gamma \to G$ can be verified by computing the values of a finite number (precisely $\binom{n}{2} + \binom{n}{3} = \frac{n^3-n}{6}$) of functions of the n matrices $\rho(\bar{\epsilon}_j) \in SL(2,\mathbb{C})$, $j = 1, ..., n$. Again, note that Theorem 4.2 is true for any algebraically closed field of characteristic 0.

Above, the criterion for irreducibility is written in terms of conditions for pairs and for triples of $SL(2,\mathbb{C})$ matrices. However, when working with representations of a free group F_n, the most important conditions are the ones for pairs because of the next result, which also shows that the condition $\nu_j = 0$ can be easily removed in the irreducible case.

Proposition 4.3. *Let $\rho : F_n \to G$ be an irreducible representation. Then, there exists a choice of generators of F_n such that the corresponding n-matrix satisfies $\sigma_{12} \neq 0$ and $\nu_1 \neq 0$.*

Proof. First note that choosing a new set of generators of F_n is equivalent to performing an automorphism of F_n. Hence, we will find such an automorphism which, upon acting on the n-matrix A associated with the irreducible ρ, will verify $\sigma_{12} \neq 0$ and $\nu_1 \neq 0$. If some $\sigma_{jk} \neq 0$ we can just permute the indices to obtain $\sigma_{12} \neq 0$. So, suppose that all σ_{jk} are zero and let (A_j, A_k, A_l) be a stable triple, so that $\Delta_{jkl} \neq 0$. Permute the generators again so that the triple becomes (A_1, A_2, A_3) and assume this triple is already in the form (2.7) with $e_1 b_2 c_3 \neq 0$ and $e_2 e_3 + b_2 c_3 = 0$. Then, perform the shift automorphism $\epsilon_1 \mapsto \epsilon_1 \epsilon_3$ of F_n, which corresponds to $A_1 \mapsto A_1' = A_1 A_3$ and an easy computation shows $\sigma_{12} \mapsto \sigma_{12}' := d_1 b_2 c_3 (d_1 b_2 c_3 + (a_1 a_3 - d_1 d_3) e_2)$. Since none of the factors d_1, b_2, c_3, e_2 and a_3 can be zero, the condition $e_2 e_3 + b_2 c_3 = 0$ implies that $\sigma_{12}' \neq 0$. Similarly, using the shift automorphism $A_1 \mapsto A_1 A_2^k$, for some $k \in \mathbb{Z}$, we end up with $\nu_1 \neq 0$. □

Consider now the stable quotient X_n^{st}/G, where $X_n^{\mathrm{st}} \subset X_n$ is the subset of stable points in X_n. As mentioned in the introduction, this is a geometric quotient and has the structure of an affine algebraic variety. Because of the identification between X_n^{st}/G and $Hom(F_n, G)^{\mathrm{irr}}/G$, where $Hom(F_n, G)^{\mathrm{irr}}$ is the subset of irreducible representations in $Hom(F_n, G)$, it is not difficult to show that X_n^{st}/G is nonsingular, and is therefore a complex analytic manifold of dimension $3n-3$ (see for example [Gu]). Also, when the finitely generated group $\Gamma = \pi_1 S$ is the fundamental group of a surface S of genus $g > 1$, the space of conjugacy classes of irreducible representations $Hom(\pi_1 S, G)^{\mathrm{irr}}/G$ can be given the structure of a complex manifold (of dimension $6g-6$), and this can be interpreted as the space of irreducible flat $SL(2,\mathbb{C})$ bundles on S ([Gu]).

In general, when the finitely generated group Γ is the fundamental group of a manifold M, it is clear that a representation $\rho : \Gamma \to G$ will define a flat rank 2 vector bundle E_ρ over M with trivial determinant. If M is an algebraic variety, one can consider moduli spaces for these bundles, and in particular, the moduli spaces of *stable* and *semistable* vector bundles on compact Riemann surfaces are well known.

In this context, one might ask what is the relationship between stability of the holomorphic vector bundle E_ρ and the stability of the n-matrix $A \in X_n$ associated with ρ. This relation is

simple in one direction. It is a general fact that if E_ρ is stable then ρ is irreducible, so A is stable, by what we saw above. However, the converse is not true and there are irreducible representations giving rise to unstable vector bundles. An example of such a vector bundle on a genus $g > 1$ Riemann surface S is the following. Let L be a degree $g-1$ line bundle whose square is the canonical bundle of S. Then the unique (up to isomorphism) indecomposable vector bundle which is an extension of the form

$$0 \to L \to E \to L^{-1} \to 0,$$

is associated to an irreducible representation $\rho : \pi_1 S \to SL(2,\mathbb{C})$. Actually, ρ is a Schottky representation, as it factors through a representation $\tilde{\rho} : F_g \to G$ of a free group of rank g, for a certain natural projection $\pi_1 S \to F_g$, and $\tilde{\rho}$ defines a Schottky group in $PSL(2,\mathbb{C})$ uniformizing S (see [Fl]). Therefore, the corresponding matrix A is stable, although $E_\rho = E$ is clearly unstable as a vector bundle. Let $Hom(\pi_1 S, G)^{\mathrm{st}}/G$ be the space of conjugacy classes of representations ρ such that E_ρ is a stable vector bundle on S. Since under the map $\rho \mapsto E_\rho$, the space $Hom(\pi_1 S, G)^{\mathrm{irr}}/G$ is the parameter space for a holomorphic family of vector bundles over S, by a result of Narasimhan and Seshadri (see [NS], Thm. 3), we conclude the following.

Proposition 4.4. *The complement of $Hom(\pi_1 S, G)^{st}/G$ inside $Hom(\pi_1 S, G)^{irr}/G$ is a nonempty analytic subset.*

The characterization of this analytic subset in terms of the geometry of S seems to be a difficult open problem.

5. The Magnus trace map T_n

In this section, we will study the Magnus trace map T_n defined in the introduction

$$\begin{array}{rcl} T_n : X_n & \to & \mathbb{C}^{3n-3} \\ A & \mapsto & (t_1, t_2, t_{12}, t_3, t_{13}, t_{23}, ..., t_k, t_{1k}, t_{2k}, ..., t_n, t_{1n}, t_{2n}) \end{array}$$

and determine when a given point in $\mathbb{C}^{3n-3}$ determines a finite (and nonzero) number of orbits of the action of $G = SL(2,\mathbb{C})$ on $X_n = G^{\times n}$. Recall the definitions (1.2) of σ_{jk} and of the Fricke discriminant Δ_{jkl}. One of the results in [M] states that

Theorem 5.1. (Magnus [M]) *Let $n \geq 4$ and $A_1, A_2, A_3 \in SL(2,\mathbb{C})$ be three fixed matrices verifying $\sigma_{12} \neq 0$ and $\Delta_{123} \neq 0$. Let $\mathbf{p} = (t_1, t_2, t_{12}, t_3, t_{13}, t_{23}) \in \mathbb{C}^6$. Then given any $\mathbf{q} \in \mathbb{C}^{3n-9}$ there exist $A_4, ..., A_n$ such that*

$$T_n(A_1, ..., A_n) = (\mathbf{p}, \mathbf{q}).$$

The number of solutions $(A_4, ..., A_n)$ is bounded by 2^{n-2}. □

This result could suggest that the image of T_n does not intersect the sets in the image corresponding to the conditions $\sigma_{12} = 0$ and $\Delta_{123} = 0$, as these are defined by polynomials in the variables t_j and t_{jk}. It turns out that the Fricke discriminant condition is not really necessary, and we only need to prevent σ_{12} from being zero.

Theorem 5.2. *Let $n \geq 2$ and $A_1, A_2 \in SL(2,\mathbb{C})$ be fixed matrices with $\sigma_{12} \neq 0$, and let $\mathbf{r} = (t_1, t_2, t_{12})$. Then given any $\mathbf{s} \in \mathbb{C}^{3n-6}$ there exist $A_3, ..., A_n$ such that*

$$T_n(A_1, ..., A_n) = (\mathbf{r}, \mathbf{s}).$$

In proving Theorem 5.2, we will use systematically the transposition invariant forms of a pair of matrices A_1, A_2 verifying $\sigma_{12} \neq 0$, given in Proposition 2.12. We will also regard elements of $SL(2,\mathbb{C})$ as complexified $SU(2)$ matrices, or as complexified quaternions with unit norm, writing them in the form

$$A_j = \begin{pmatrix} \alpha_j + i\beta_j & \gamma_j + i\delta_j \\ -\gamma_j + i\delta_j & \alpha_j - i\beta_j \end{pmatrix} \in SL(2,\mathbb{C}), \quad j = 1, ..., n, \tag{5.1}$$

with

$$\alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbb{C}, \text{ and } \quad \alpha_j^2 + \beta_j^2 + \gamma_j^2 + \delta_j^2 = 1. \tag{5.2}$$

For matrices in $G = SL(2,\mathbb{C})$, and using the parameterizations of A_j as in (5.1), the following is an immediate consequence of the proof of Proposition 2.12.

Proposition 5.3. *Let $A = (A_1, A_2) \in X_2$ verify $\sigma_{12} \neq 0$. If $\nu_1 \neq 0$, then A is similar to a pair $B = (B_1, B_2)$ of the form*

$$B_1 = \begin{pmatrix} \alpha_1 + i\beta_1 & 0 \\ 0 & \alpha_1 - i\beta_1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \alpha_2 + i\beta_2 & i\delta_2 \\ i\delta_2 & \alpha_2 - i\beta_2 \end{pmatrix},$$

with $\alpha_1^2 + \beta_1^2 = 1$, $\alpha_2^2 + \beta_2^2 + \delta_2^2 = 1$ and $\delta_2^2 = \frac{\sigma_{12}}{2\nu_1} \neq 0$. In case $\nu_1 = 0$ and $\nu_2 \neq 0$, the situation is completely analogous switching B_1 with B_2. If $\nu_1 = \nu_2 = 0$, then A is similar to a pair $B = (B_1, B_2)$ of the form

$$B_1 = \begin{pmatrix} \alpha_1 + \lambda & i\lambda \\ i\lambda & \alpha_1 - \lambda \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \alpha_2 - \lambda & i\lambda \\ i\lambda & \alpha_2 + \lambda \end{pmatrix},$$

with $\alpha_1^2 = \alpha_2^2 = 1$ and $\lambda^4 = \frac{\sigma_{12}}{16} \neq 0$. □

To adapt our notation to the trace map T_n, let us denote the components of an arbitrary element $\mathbf{z}$ of the range $\mathbb{C}^{3n-3}$ by

$$\mathbf{z} = (z_1, z_2, z_{12}, ..., z_k, z_{1k}, z_{2k}, ..., z_n, z_{1n}, z_{2n}), \tag{5.3}$$

and define

$$\begin{aligned} \nu_j(\mathbf{z}) &= \frac{z_j^2}{2} - 2, \\ \sigma_{jk}(\mathbf{z}) &= z_j^2 + z_k^2 + z_{jk}^2 - z_j z_k z_{jk} - 4. \end{aligned}$$

In this way, given any polynomial $p(\mathbf{z})$ in the coordinates of $\mathbf{z}$, its pullback under T_n will be the same polynomial in the variables $(t_1, t_2, t_{12}, ..., t_n, t_{1n}, t_{2n})$. To simplify the notation, we will sometimes write these polynomial functions without reference to the variables. Consider the following Zariski closed subsets of X_n and of $\mathbb{C}^{3n-3}$, respectively,

$$\begin{aligned} U_{12} &= \{A \in X_n : \sigma_{12}(A) = 0\} \\ Z_{12} &= \{\mathbf{z} \in \mathbb{C}^{3n-3} : \sigma_{12}(\mathbf{z}) = 0\}, \end{aligned}$$

so that $T_n(U_{12}) = Z_{12}$. It is immediate that Theorem 5.2 above can be restated as

Theorem 5.4. *The algebraic map $T_n : X_n \setminus U_{12} \to \mathbb{C}^{3n-3} \setminus Z_{12}$ is surjective for $n \geq 2$.*

Proof. We need to show that, for any given $\mathbf{z}$ as in (5.3) verifying $z_1^2 + z_2^2 + z_{12}^2 - z_1 z_2 z_{12} \neq 4$, there is a set of n matrices

$$A_j = \begin{pmatrix} \alpha_j + i\beta_j & \gamma_j + i\delta_j \\ -\gamma_j + i\delta_j & \alpha_j - i\beta_j \end{pmatrix} \in SL(2,\mathbb{C}), \quad j = 1, ..., n,$$

such that $\mathsf{tr}(A_j) = z_j$ and $\mathsf{tr}(A_j A_k) = z_{jk}$, for all coordinates of $\mathbf{z}$. Let us start with the case when $\nu_1(\mathbf{z}) \neq 0$ which means $z_1 \neq \pm 2$ and corresponds under T_n to $\nu_1(A) \neq 0$. Then, using Proposition 5.3, if there is a solution, there is one in the form

$$A_1 = \begin{pmatrix} \alpha_1 + i\beta_1 & 0 \\ 0 & \alpha_1 - i\beta_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \alpha_2 + i\beta_2 & i\delta_2 \\ i\delta_2 & \alpha_2 - i\beta_2 \end{pmatrix}, \tag{5.4}$$

$$A_k = \begin{pmatrix} \alpha_k + i\beta_k & \gamma_k + i\delta_k \\ -\gamma_k + i\delta_k & \alpha_k - i\beta_k \end{pmatrix}, \quad k = 3, ..., n.$$

Then, we need to solve

$$\begin{aligned} z_k &= 2\alpha_k, \qquad k = 1, ..., n, \\ z_{1k} &= \mathsf{tr}(A_1 A_k) = 2(\alpha_1\alpha_k - \beta_1\beta_k), \qquad k = 2, ..., n, \\ z_{2k} &= \mathsf{tr}(A_2 A_k) = 2(\alpha_2\alpha_k - \beta_2\beta_k - \delta_2\delta_k), \qquad k = 3, ..., n, \end{aligned} \tag{5.5}$$

and an explicit solution is obtained by setting $\alpha_k = z_k/2$, for $k = 1, ..., n$ and

$$\beta_1 = \sqrt{1 - \alpha_1^2}, \qquad \beta_k = \frac{z_1 z_k - \frac{1}{2} z_{1k}}{\beta_1}, \qquad k = 2, ..., n,$$

$$\delta_2 = \sqrt{q_{22}}, \quad \delta_k = \frac{q_{2k}}{\delta_2}, \qquad k = 3, ..., n, \quad \text{and} \tag{5.6}$$

$$\gamma_k = \frac{\sqrt{q_{22}q_{kk} - q_{2k}^2}}{\delta_2}, \qquad k = 3, ..., n,$$

where we have used the abbreviations $q_{kk} = 1 - \alpha_k^2 - \beta_k^2$ and $q_{jk} = \alpha_j\alpha_k - \beta_j\beta_k - \frac{1}{2}z_{jk}$, if $j \neq k$. The denominators β_1 and δ_2 are both nonzero because of our assumptions on $\nu_1(\mathbf{z}) = -2\beta_1^2$ and $\sigma_{12}(\mathbf{z}) = -4\beta_1^2\delta_2^2 = 2\nu_1\delta_2^2$. Observe that, after assuming A_1 diagonal and $A_2 = A_2^T$, the above are the only solutions, and different solutions correspond to different choices of square roots in the above expressions. Also note that the Fricke discriminant appears as $\Delta_{12k} = 16\beta_1^2\delta_2^2\gamma_k^2$.

When $\nu_1(\mathbf{z}) = \nu_2(\mathbf{z}) = 0$, from Proposition 5.3, we may use the parametrization

$$A_1 = \begin{pmatrix} \alpha_1 + \lambda & i\lambda \\ i\lambda & \alpha_1 - \lambda \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \alpha_2 - \lambda & i\lambda \\ i\lambda & \alpha_2 + \lambda \end{pmatrix}, \qquad \lambda \neq 0 \tag{5.7}$$

$$A_k = \begin{pmatrix} \alpha_k + i\beta_k & \gamma_k + i\delta_k \\ -\gamma_k + i\delta_k & \alpha_k - i\beta_k \end{pmatrix}, \quad k = 3, ..., n,$$

where α_1 and α_2 are square roots of 1. Now, the equations $T_n(A) = \mathbf{z}$ become

$$\begin{aligned} z_k &= 2\alpha_k \\ z_{1k} &= 2\alpha_1\alpha_k + 2\lambda(i\beta_k - \delta_k) \\ z_{2k} &= 2\alpha_2\alpha_k + 2\lambda(-i\beta_k - \delta_k), \qquad k = 3, ..., n \end{aligned} \tag{5.8}$$

which one can easily solve for α_k, β_k, and δ_k. The variables γ_k are then obtained from the normalization (5.2). As before, the solutions in this form will be in finite number. □

As remarked, the proof of Theorem 5.4 implies that any point $\mathbf{z} \in \mathbb{C}^{3n-3} \setminus Z_{12}$ determines an orbit up to a finite ambiguity. More concretely, we describe all the solutions as follows.

Theorem 5.5. *Let $\mathbf{z} \in \mathbb{C}^{3n-3} \setminus Z_{12}$ and $A = (A_1, A_2, ..., A_n)$ be a solution of $T_n(A) = \mathbf{z}$, such that A_1 and A_2 are invariant under transposition. Then $T_n^{-1}(z)$ is the G orbit of the finite set*

$$\{(A_1, A_2, B_3, ..., B_n) : \ B_k = A_k \ \text{or} \ B_k = A_k^T\} \subset X_n \tag{5.9}$$

of cardinality $\leq 2^{n-2}$.

Proof. First, let $\mathbf{z} \in \mathbb{C}^{3n-3} \setminus Z_{12}$ verify $z_1 \neq \pm 2$, and let $\mathcal{S}_1$ be the space of solutions of $T_n(A) = \mathbf{z}$ with A_1 diagonal and $A_2^T = A_2$. From the proof of Theorem (5.4) and from Propoposition (5.3), any $A \in \mathcal{S}_1$ is obtained from formulas (5.4) and (5.6), for a certain choice of square roots of β_1, δ_2 and γ_3,..., γ_n. Changing all the signs of $\beta_1, ..., \beta_n, \gamma_3, ..., \gamma_n$ simultaneously, gives a well defined map $\sigma_1 : \mathcal{S}_1 \to \mathcal{S}_1$. Similarly, let $\sigma_2 : \mathcal{S}_1 \to \mathcal{S}_1$ be the operation of changing simultaneously the signs of $\delta_2, ..., \delta_n, \gamma_3, ..., \gamma_n$. The signs of each γ_k, $k = 3, ..., n$ can be changed independently of the rest, giving maps $\sigma_k : \mathcal{S}_1 \to \mathcal{S}_1$, $k = 3, ..., n$. For matrices in the form (5.1), it is clear that conjugation by $g = (0, i, i, 0)$ changes the triple of vectors (β, γ, δ) into $(-\beta, -\gamma, \delta)$, and conjugation by $g = (i, 0, 0, -i)$ maps (β, γ, δ) into $(\beta, -\gamma, -\delta)$ (naturally any conjugation fixes the vector α). Therefore, σ_1 and σ_2 act trivially on the space $\mathcal{S}_1/G$. Since transposition of a single A_k changes the sign of a single γ_k, $k = 3, ..., n$, the orbits are as in (5.9). Similarly, to treat the case $z_1 = \pm 2$, let $\mathcal{S}_0$ be the space of solutions of $T_n(A) = \mathbf{z}$ of the form (5.7), (5.8). As before, define $\sigma_0 : \mathcal{S}_0 \to \mathcal{S}_0$ by changing simultaneously the signs of $\lambda, \beta_1, ..., \beta_n, \delta_1, ..., \delta_n$. By the same reason as before, this acts trivially on $\mathcal{S}_0/G$, so also in this case, the set of solutions is given by (5.9). □

We have thus finished the proof of Theorem 1.3.

Appendix A. Reconstruction of matrices from traces

In this Appendix, we briefly indicate the generic reconstruction of (the $SL(2,\mathbb{C})$ orbit of) general n-tuples of 2×2 matrices $A \in V_n$ from a minimal number of traces given by the analogous trace map

$$\begin{aligned} \hat{T}_n : V_n &\rightarrow \mathbb{C}^{4n-3} \\ A &\mapsto (t_1, t_{11}, t_2, t_{22}, t_{12}, t_3, t_{33}, t_{13}, t_{23}, ..., t_n, t_{nn}, t_{1n}, t_{2n}). \end{aligned} \tag{A.1}$$

The formulas are identical to the ones in Theorem 5.4, when written only in terms of the functions $\tau_{jk} : \mathbb{C}^{4n-3} \rightarrow \mathbb{C}$, given by

$$\tau_{jk}(\mathbf{z}) = z_{jk} - \frac{1}{2} z_j z_k, \qquad j,k \in \{1,n\},$$

except that now we allow coordinates z_{jk} with $j = k$ and, writing A_j in the form (5.1) we don't require the normalization condition (5.2). For instance, for $\sigma_{12}(\mathbf{z}) \neq 0$, and $\nu_1(\mathbf{z}) \neq 0$, the following matrices form a solution to $\hat{T}_n(A) = \mathbf{z}$.

$$A_1 = \begin{pmatrix} \alpha_1 + i\beta_1 & 0 \\ 0 & \alpha_1 - i\beta_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \alpha_2 + i\beta_2 & i\delta_2 \\ i\delta_2 & \alpha_2 - i\beta_2 \end{pmatrix},$$

$$A_k = \begin{pmatrix} \alpha_k + i\beta_k & \gamma_k + i\delta_k \\ -\gamma_k + i\delta_k & \alpha_k - i\beta_k \end{pmatrix}, \quad k = 3, ..., n,$$

with

$$\beta_1 = \sqrt{-\frac{\tau_{11}}{2}} \qquad \beta_k = -\frac{\tau_{1k}}{2\beta_1}, \qquad k = 2, ..., n,$$

$$\delta_2 = \sqrt{\frac{\sigma_{12}}{2\tau_{11}}}, \qquad \delta_k = \frac{\tau_{2k}\tau_{11} - \tau_{12}\tau_{1k}}{2\delta_2\tau_{11}}, \qquad , k = 3, ..., n,$$

$$\gamma_k = \sqrt{-\frac{\Delta_{12k}}{4\sigma_{12}}}, \qquad k = 3, ..., n,$$

where the 'Fricke discriminant', as a function of $\tau_{jk}(\mathbf{z})$, is given by

$$\Delta_{12k}(\mathbf{z}) = 2(\tau_{12}^2\tau_{kk} + \tau_{1k}^2 t_{22} + \tau_{2k}^2\tau_{11} - 2\tau_{12}\tau_{1k}\tau_{2k} - \tau_{11}\tau_{22}\tau_{kk}),$$

in agreement with (2.5). The case with $z_1, z_2 \in \{-2, 2\}$ can be treated similarly.

Appendix B. Examples with $T_n^{-1}(\mathbf{z}) = \emptyset$.

Here, we show that the image of T_n, for $n \geq 4$, does not contain certain points $\mathbf{z} \in \mathbb{C}^{3n-3}$. From Theorem 5.2, if the equation $T_n(A) = \mathbf{z}$ has no solution $A \in X_n$, then $\sigma_{12}(\mathbf{z}) = 0$. Take, for example, any $\mathbf{z}$ verifying $z_1 = z_2 = z_{12} = 2$, and for which there are two indices $k, l \notin \{1, 2\}$ such that

$$(z_{1k} - z_k)(z_{2l} - z_l) \neq (z_{1l} - z_l)(z_{2k} - z_k). \tag{B.1}$$

Then, without loss of generality A can be taken in the form:

$$A_1 = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix},$$

$$A_j = \begin{pmatrix} \alpha_j + i\beta_j & b_j \\ c_j & \alpha_j - i\beta_j \end{pmatrix}, \quad j = 3, ..., n.$$

Suppose $T_n(A) = \mathbf{z}$. Then, we should have, for all $j = 3, ..., n$,

$$\begin{aligned} z_j &= 2\alpha_j \\ z_{1j} &= 2\alpha_j + b_1 c_j \\ z_{2j} &= 2\alpha_j + b_2 c_j, \end{aligned} \tag{B.2}$$

In the case that b_1 and b_2 are nonzero, this implies

$$c_k = \frac{z_{1k} - z_k}{b_1} = \frac{z_{2k} - z_k}{b_2}, \qquad c_l = \frac{z_{1l} - z_l}{b_1} = \frac{z_{2l} - z_l}{b_2},$$

which is impossible by our hypothesis (B.1). Similarly, if $b_1 = 0$, equations (B.2) imply $z_{1k} = z_k$ and $z_{1l} = z_l$, contradicting again (B.1). The same happens if $b_2 = 0$.

More generally, a similar situation occurs in the case that $\sigma_{12}(\mathbf{z}) = 0$ but, for example, $\nu_1(\mathbf{z})$ is nonzero. In this case, without loss of generality (see lemma 2.8), we have the parametrization

$$A_1 = \begin{pmatrix} \alpha_1 + i\beta_1 & 0 \\ 0 & \alpha_1 - i\beta_1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \alpha_2 + i\beta_2 & 1 \\ 0 & \alpha_2 - i\beta_2 \end{pmatrix}, \quad \text{or} \quad A_2 = \begin{pmatrix} \alpha_2 + i\beta_2 & 0 \\ 0 & \alpha_2 - i\beta_2 \end{pmatrix},$$

$$A_k = \begin{pmatrix} \alpha_k + i\beta_k & b_k \\ c_k & \alpha_k - i\beta_k \end{pmatrix}, \quad k = 3, ..., n,$$

Assuming A_2 non-diagonal, we need to solve the equations, for every $k = 3, ..., n$.

$$\begin{aligned} z_k &= 2\alpha_k \\ z_{1k} &= \mathsf{tr}(A_1 A_k) = 2(\alpha_1\alpha_k - \beta_1\beta_k) \\ z_{2k} &= \mathsf{tr}(A_2 A_k) = 2(\alpha_2\alpha_k - \beta_2\beta_k) + c_k. \end{aligned} \tag{B.3}$$

One finds that $\beta_1 = \sqrt{-\frac{\nu_1(\mathbf{z})}{2}} \neq 0$, $\beta_2 = \sqrt{-\frac{\nu_2(\mathbf{z})}{2}}$ and

$$\begin{aligned} \beta_k &= \frac{2\alpha_1\alpha_k - z_{1k}}{2\beta_1} = -\frac{\tau_{1k}(\mathbf{z})}{2\beta_1} \\ c_k &= \frac{\beta_1\tau_{2k}(\mathbf{z}) - \beta_2\tau_{1k}(\mathbf{z})}{\beta_1} \end{aligned}$$

If $c_k \neq 0$, then the b_k can be found using the normalization $\alpha_k^2 + \beta_k^2 - b_k c_k = 1$. However, when $\beta_1\tau_{2k}(\mathbf{z}) = \beta_2\tau_{1k}(\mathbf{z})$, for some $k \neq 1, 2$, in the above expression we obtain $c_k = 0$ which implies that A_2 has to be in diagonal form, and then c_k disappears from equation (B.3), so we need to have $\beta_1\tau_{2j}(\mathbf{z}) = \beta_2\tau_{1j}(\mathbf{z})$ for all $j \neq 1, 2$. Obviously, this does not hold in general. Note also that these computations, combined with the previous case $z_1 = \pm 2$, allows one to reprove surjectivity for $n = 3$ (in this case, $c_3 = 0$ is exactly equivalent to the necessary condition $\beta_1\tau_{23}(\mathbf{z}) = \beta_2\tau_{13}(\mathbf{z})$).

References

[CS] M. Culler and P. B. Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. Math. **117** (1983) 109-146.

[Fl] C. Florentino, *Schottky uniformization and vector bundles over Riemann surfaces*, Manuscripta Math. **105** (2001), 69-83.

[FK] R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Functionen*, Vol. 1, 365-370. Leipzig, B. G. Teubner 1987.

[Fr] S. Friedland, *Simultaneous similarity of matrices*, Adv. in Math. **50** (1983) 189-265.

[Gi] D. Gieseker, *Geometric invariant theory and the moduli of bundles*, in Gauge Theory and the Topology of Four-Manifolds, AMS, 1998.

[Go] W. Goldman, *An exposition of results of Fricke and Vogt*, Preprint, math.GM/0402103.

[GM] F. González-Acuña and J. M. Montesinos-Amilibia, *On the character variety of group representations in* $SL(2,\mathbb{C})$ *and* $PSL(2,\mathbb{C})$, Math. Zeit. **214** (1993) 627-652.

[Gu] R. Gunning, Lectures on Vector Bundles over Riemann Surfaces, Princeton U.P, 1967; *Analytic structures on the space of flat vector bundles over a compact Riemann surface*, LNM **185**, Springer, 1971, 47-62.

[H] R. Horowitz, *Characters of free groups represented in the two-dimensional special linear group*, Comm. Pure Appl. Math. **25** (1972), 635-649.

[M] W. Magnus, *Rings of Fricke characters and automorphism groups of free groups*, Math. Zeit. **170** (1980) 91-103.

[MFK] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory*, 3rd Edition, Springer Verlag, 1994.

[NS] M. S. Narasimhan, C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) **82** (1965) 540-567.

[P] C. Procesi, *The Invariant theory of* $n \times n$ *matrices*, Adv. in Math. **19** (1976) 306-381.

[V] H. Vogt, *Sur les invariants fondamentaux des équations différentielles linéaires du second ordre*, Ann. de l'Éc. Norm, Serie VI, **3** Suppl. (1889) 3-72. (Thèse, Paris.)